Department of Electrical and Computer Engineering
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Senior Project Design

Electron Matter-wave Tractor Beams: Study & Simulation of Quantum-Mechanical Stress and Force

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Declaration

This is to declare that no part of this report or the project has been previously submitted elsewhere for the fulfillment of any other degree or program. Proper acknowledgement has been provided for any material that has been taken from previously published sources in the bibliography section of this report.

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Approval

The Senior Design Project entitled "Electron Matter-wave Tractor Beams: Study & Simulation of Quantum-Mechanical Stress and Force" by Md. Arifur Rahman, Md. Mahbub Alam Arafin and Jahidul slam has been accepted as satisfactory and approved for partial fulfillment of the requirement of BS in EEE degree program.

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Abstract

Classical force equations describing the macroscopic universe have been studied and applied extensively throughout the past three centuries. However, the study & simulation of quantum-mechanical stress & force in the microscopic domain remain scantily explored. We attempt to accelerate the research of microscopic scenarios involving quantum-mechanical forces and successfully devise a COMSOL Multiphysics simulation setup and derive all the corresponding mathematical formulations to observe a wide array of quantum-mechanical phenomena, in particular, the simulation of electron matter-wave tractor beams in COMSOL Multiphysics 6.0.

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Chapter 1: Introduction

1.1 Optical Pulling Force and Tractor Beams

Johannes Kepler hypothesized in 1619 that the pressure of sunlight blasted back a comet's tail, allowing it to constantly point away from the Sun. However, this was solely an assertion at the time because science and technology just weren't evolved enough to evaluate such an issue. Then, in 1873, James Clarke Maxwell theoretically showed that light waves indeed impose pressure, writing, "In a medium in which waves are propagated, there is a pressure in the direction of normal to the waves, and numerically equivalent to the energy in a unit of volume.". Einstein demonstrated that light has wave particle duality, which means that it behaves as both a wave and a particle. This means that light incident on any surface will cause bombardment of these particles over the surface, exerting a force on the particle that will act in the same direction as the propagation of light. As we can see, individuals had a strong understanding of light's capacity to impose radiation pressure. This was the standard worldview, which adhered to all basic physics principles such as momentum conservation. Afterward, in 1970, the discovery of optical tweezers opened up a new area of research in which scientists started working on manipulating different parameters of the incident light to exert non-conservative force over the radiated object, with the illuminated object having its resultant force in the direction opposite to the wave propagation of the incident beam, which scientists dubbed optical tractor beam. The optical tweezer evolved into a strong instrument for manipulating micro-objects and biomolecules that was utilized in several disciplines of research for biomolecule analysis, sorting, trapping, and binding of particles on the microscale or nanoscale, and so on. Such optical manipulation approaches have evolved into valuable research tools, with applications spanning from biology and biomedicine to microfluidics and colloidal science. Recently, there has been an increase in interest in levitation dynamics in optomechanical systems, including passive and active feedback techniques, for attaining quantum mechanical superposition, ground-state cooling, and mechanical system coherent manipulation.

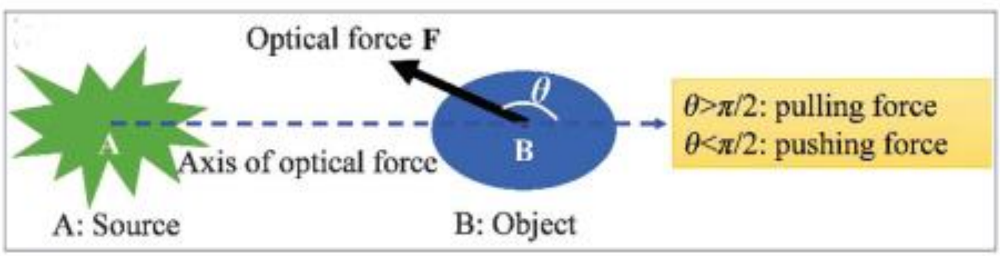

Fig 1.1: Visual representation of optical pulling force [1].

Light-matter interaction has been a fascinating subject of significant analytical and experimental investigation since the development of Maxwell's electromagnetic wave theory. Researchers have been researching the optical pressures produced on particles triggered by incoming light waves for several decades. Light scattering occurs when light interacts with materials and releases energy from the particle. The divergence of the Maxwell stress tensor provides a good measure of the total optical forces operating on a particle [2].

Tractor beams are beams that can cause pulling. A paradoxical phenomenon is exploited, in which all objects are drawn towards the source. When an incident beam collides with an object or particle, light matter interaction produces optical force, and there are several methods for achieving this force. They may be divided into four groups [1]. They are:

1. Using structured beams.
2. Using objects with exotic structures and parameters.
3. Using a structured background that supports special modes.
4. Using the photophoretic force that result from light absorption.

Here is a brief overview of these four categories for achieving optical pulling force.

Structured beams:

Using a structured light beam beyond the plane wave and the paraxial Gaussian beam we can achieve optical pulling force [3] [4]. There are several different specific configurations in this mechanism. For example, the first one is using a single-structured diffraction-free beam, such as the Bessel beam and solenoid beam [5] [6]. The other one is using the interferences of two or more structured waves, such as two Bessel beams, two Gaussian beams, or multiple plane waves [5].

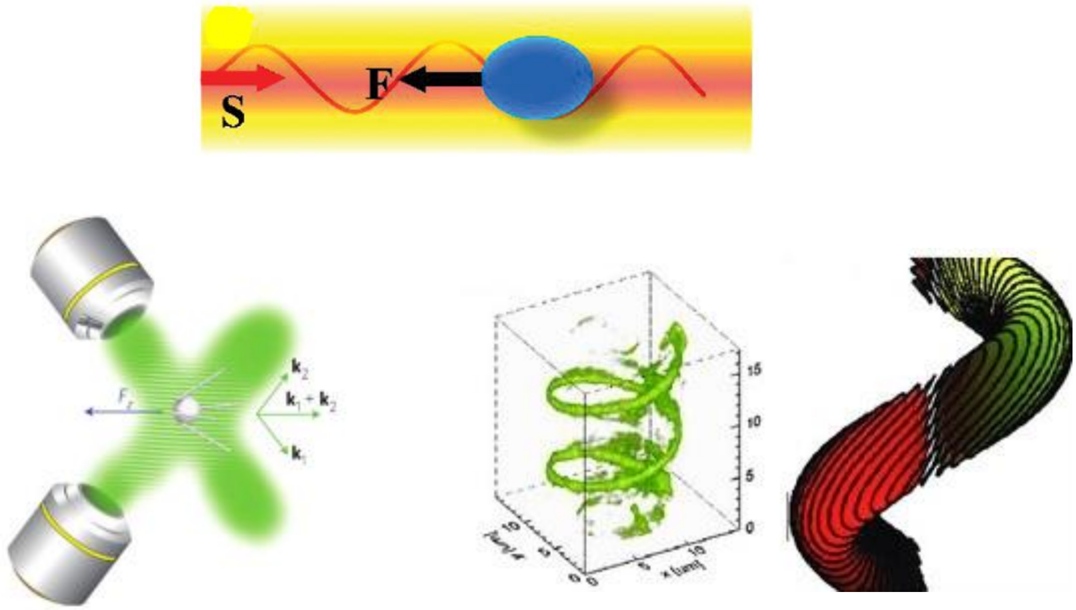

Fig 1.2: Achieving optical pulling force using structured light beams [1].

Object Properties:

Except for using one or more structured light beams, optical pulling force is also possible to achieve using an object with proper optical features. For example, optical gain object such as, slabs, spheres, and deep subwavelength structures [7] [8] [9]. Optical pulling force is also possible for lossy objects [13]. Again we can achieve optical pulling force by using the interaction of a chiral object with chiral light and also using two coherent plane waves with counter propagation [10] [11] [12].

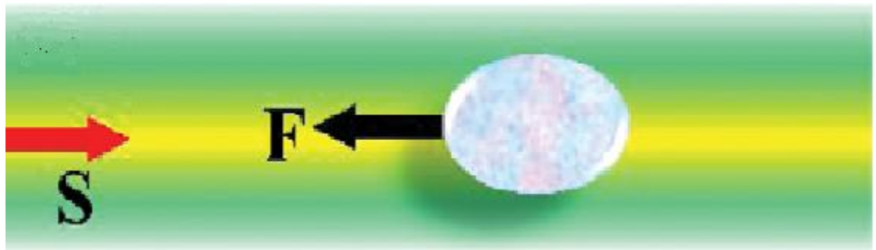

Fig 1.3: Optical pulling force by exotic optical properties [1].

Structured Background:

The background medium is too much important in the interaction of light and matter [14]. A structured background provides enriched properties for light and matter interaction, which may also greatly modulate the scattering properties and thus the optical force behavior. For example, interfacial tractor beam such as, two homogeneous backgrounds, plasmonic interface, another one is waveguide channels and waveguide with negative mode index, and the self-collimation mode in photonic crystals [15] [16] [17].

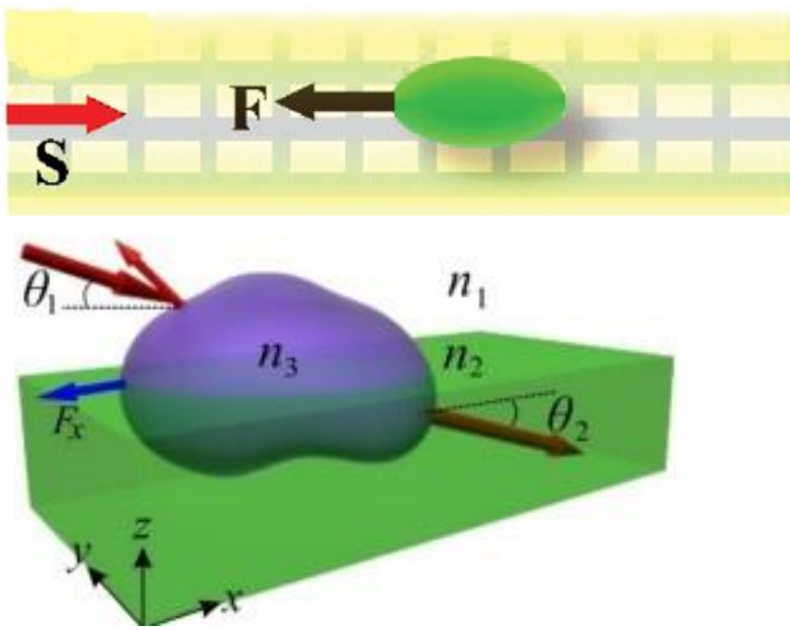

Fig 1.4: Optical pulling force by structured background [1].

Optical Pulling by Photophoresis:

Photophoretic force is induced by inhomogeneous temperature distribution on an object when it absorbs incident light and bounces off the molecules of a fluidic background asymmetrically. In fluidic (both liquid and gaseous) environments, when a laser beam illuminates an absorptive object, a temperature gradient appears on the object and bounces off the molecules of the fluidic background asymmetrically [18] [19]. As a result, the object may get a net force.

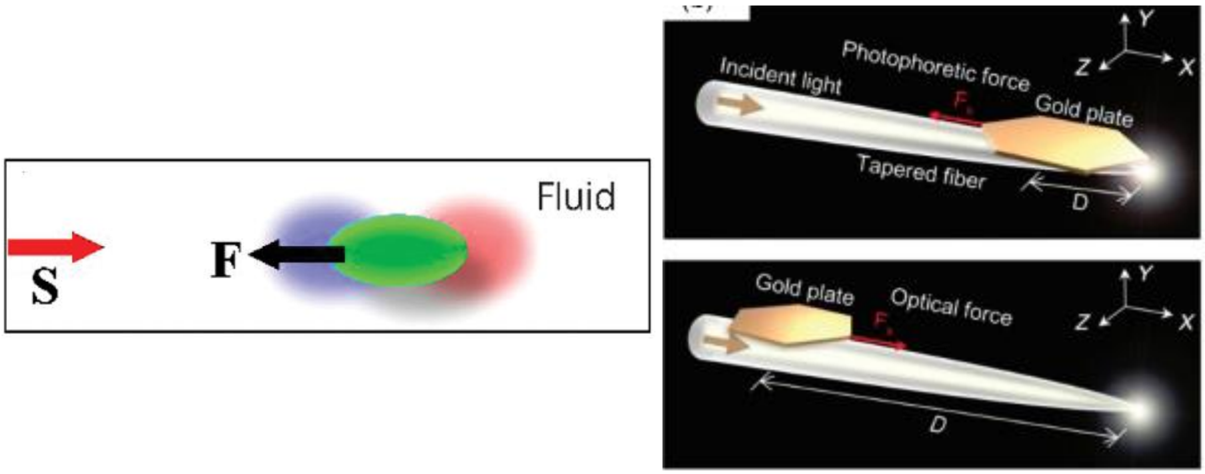

Fig 1.5: Optical pulling force assisted by photophoretic force [1].

Theoretically, the photophoretic force could be about 10^5 times larger than the direct optical force, which makes this force extremely important in giant optical manipulation [20].

In a sense the optical pulling force and tractor beam is the forefather of quantum mechanical pulling force.

1.2 Analogy between Optics & Quantum Mechanics

The following mathematical derivations have been excerpted from the supplemental material to ‘Matter-wave Tractor Beams’ in their entirety as they are very relevant to our work on matter-wave tractor beams. [21]

The radial wave function at infinity takes the form

$$\psi(r,\theta,\phi) = \psi_i(\mathbf{r}) + f(\theta,\phi)\frac{e^{ik_0 r}}{r}$$

(1.1)

Substituting the phase shift tan $\delta_l = -\frac{B_l}{A_l} = -\frac{1}{K_l}$ into

$$F_z = \frac{4\pi\hbar^2}{\mu}|A_0|^2 \sum_l (l+1)P_{l+1}(\cos\ \alpha)P_l(\cos\ \alpha)\sin^2(\delta_{l+1} - \delta_l)$$

(1.2)

We get the quantum-mechanical force:

$$F_z = \sum_l F_l(E, U_0)P_l(\cos\ \alpha)P_{l+1}(\cos\ \alpha)$$

(1.3)

Which we can rewrite as:

$$F_Z = \beta\left[F_0 + \frac{3F_1}{2}\beta^2 - \frac{F_1}{2}\right]$$

(1.4)

where $\beta = cos\alpha$ the optical force exerted by the nondiffractive beam in dipolar approximation [21]

$$F_z = \frac{k_0\beta}{2}[\mathrm{Im}\ (\alpha_e)|\mathbf{E}|^2 + \mathrm{Im}\ (\alpha_m)|\mathbf{H}|^2] - \frac{k_0^4}{3}\mathrm{Re}\ [\alpha_e\alpha_m^*(\mathbf{E}\times\mathbf{H}^*)_z]$$

(1.5)

where α_e and α_m are the polarizabilities of the spherical bead, E and H are respectively electric and magnetic field strengths, k0=ω/c is vacuum wavenumber, ω is circular frequency and c is the speed of light. Assuming the optical wave is the superposition of two plane waves with incident angles α and –α (they form the nondiffractive light beam) and choosing TE polarization, we can write the electric and magnetic fields as $\mathbf{E} = Ey\mathbf{e}y$ and $\mathbf{H} = -\beta Ey\mathbf{e}x$, respectively. By substituting the fields into the above equation, we arrive at

$$F_z = \beta\left[\frac{k_0|E_Z|^2}{2}\mathrm{Im}\,(\alpha_e) + \beta^2\frac{k_0|E_z|^2}{2}\mathrm{Im}\,(\alpha_m) - \frac{k_0^4|E_Z|^2}{3}\mathrm{Re}\,(\alpha_e\alpha_m^*)\right]$$

(1.6)

We observe equations (1.4) and (1.6) to be similar. Which implies that the interaction of quantum particle with Bessel wave function can be considered to that of spherical particles with polarizabilities in the field of two plane waves.

$$\frac{k_0|E_Z|^2}{2}\mathrm{Im}\,(\alpha_e) - \frac{k_0^4|E_z|^2}{3}\mathrm{Re}\,(\alpha_e\alpha_m^*) = F_0 - \frac{F_1}{2}, \frac{k_0|E_z|^2}{2}\mathrm{Im}\,(\alpha_m) = \frac{3F_1}{2}$$

(1.7)

Electric field Ez can be associated with the amplitude A0 and the energy E of the incident matter wave can be replaced with the optical circular frequency ω. Hence, we get the frequency-dependent polarizabilities of the magneto-dielectric particles as

$$\mathrm{Im}\,(\alpha_e) - \frac{2k_0^3}{3}\mathrm{Re}\,(\alpha_e\alpha_m^*)$$

$$= \frac{C_0(K_1(\omega) - K_0(\omega))^2}{\omega(1 + K_0^2(\omega))(1 + K_1^2(\omega))} - \frac{C_0(K_2(\omega) - K_1(\omega))^2}{\omega(1 + K_1^2(\omega))(1 + K_2^2(\omega))}$$

$$\mathrm{Im}\,(\alpha_m) = \frac{3C_0(K_2(\omega) - K_1(\omega))^2}{\omega(1 + K_1^2(\omega))(1 + K_2^2(\omega))}$$

(1.8)

where $C0$ is a constant. Since the complex values of electric and magnetic polarizabilities are not uniquely defined, we have a great variety of frequency dispersive permittivities and permeabilities that are able to mimic the behavior of matter-wave pulling forces.

Chapter 2: Methodology

2.1 Experimental Setup

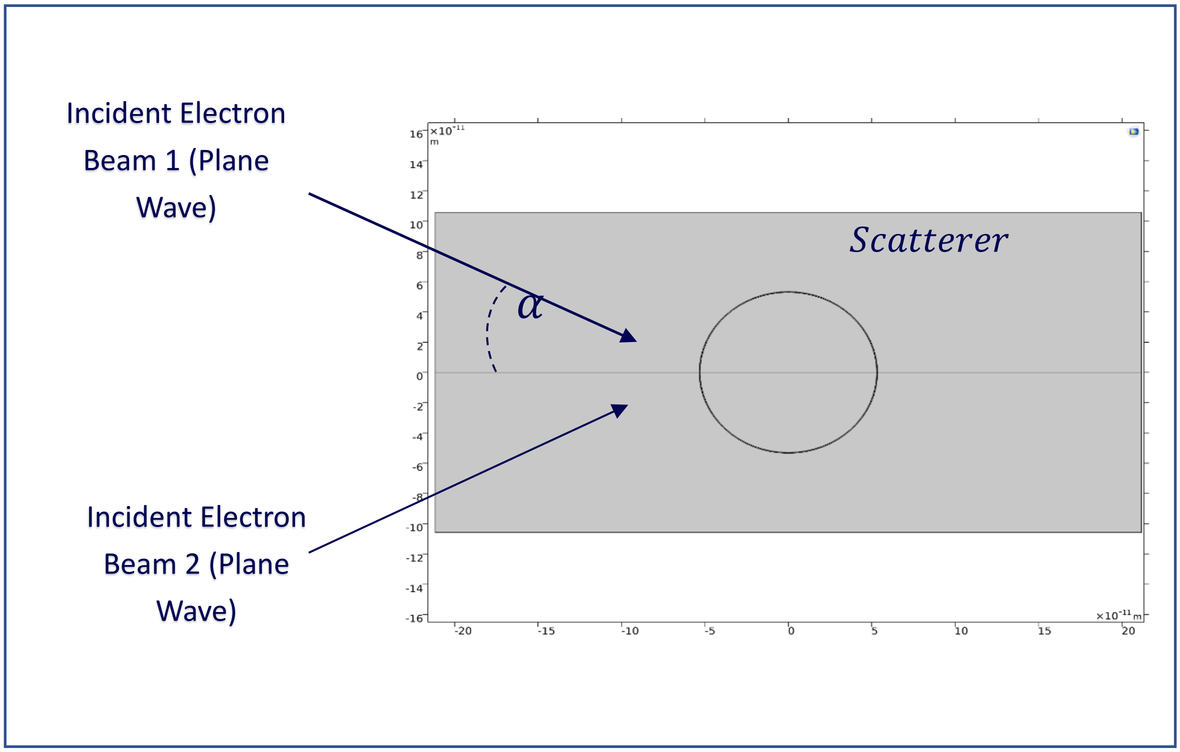

Figure 2.1: Experimental setup

We go for the simplest experimental setup possible as shown in Ref [21] with slight modification in the incident particle as electrons and the target scatterer as a Helium atom. This is due to the inherent limitation of our simulation environment.

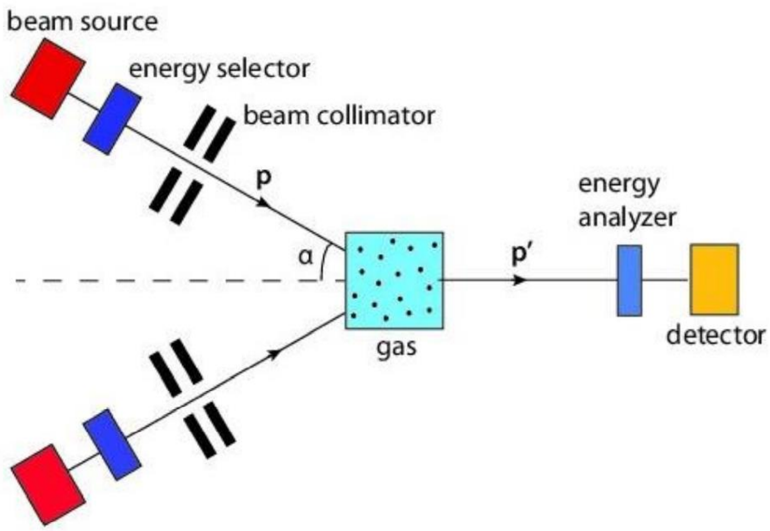

Fig. 2.2 Simplest Experimental Setup as Shown in Literature [21]

2.2 Born Approximation

Generally, in scattering theory and in particular in quantum mechanics, the Born approximation consists if taking the incident filed in place of the total filed as the driving field at each point in the scatterer. It is the perturbation method applied to scattering by an extended body. It is accurate if the scattered field is small compared to the incident filed on the scatterer. We use Born approximation to find the scattering amplitude when the kinetic energy of incident wave is large and when the potential barrier is weak.

For weak potential barrier the higher order approximation is negligible and the 1st order is sufficient. For zero order born approximation there is no scattering. So, $\psi_0(r') = 0$.

So, from the solution of Schrodinger equation we get,

$$\psi(r) = \phi_{inc} \qquad ;[where\ \phi_{inc} = e^{\iota kr}]$$

First Order Born Approximation:

For 1st order born approximation from the solution of Schrodinger equation we get,

$$\psi_1(r) = \phi_{inc} - \frac{\mu}{2\pi\hbar}\int \frac{e^{\iota k|r-r'|}}{|r-r'|} v(\overline{r'})\, \phi_{inc}(r)\, d^3r'$$

When, $\psi(r') = \phi_{inc}(r')$ the scattering amplitude is,

$$f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int e^{-\iota kr'} v(\overrightarrow{r'})\, \phi_{inc}(r)\, d^3r'$$

$$\rightarrow f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int e^{-\iota kr'} v(\overrightarrow{r'}) e^{\iota k_0 r'}\, d^3r'$$

$$\rightarrow f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int e^{\iota r'(k_0-k)}\, v(\overrightarrow{r'})\, d^3r'$$

$(k_0 - k) = q$ which is momentum transfer.

So that, scattering amplitude of general potential is,

$$f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int e^{\iota r' q}\, v\left(\overrightarrow{r'}\right)\, d^3r'$$

And differential cross section is,

$$\sigma(\theta,\phi) = \frac{\mu^2}{4\pi^2\hbar^2}\left|\int e^{\iota r' q}\, v(\overrightarrow{r'})\, d^3r'\right|^2$$

This equation is for general potential.

For spherical symmetric potential,

$$v(\overrightarrow{r'}) = v(r') \qquad ;\text{[as it is scalar]}$$

Hence,

$$f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int e^{\iota r' q}\, v(r')\, d^3r'$$

Here, $q.r' = qr'cos\theta'$ and $\int d^3r' = \int_0^\infty r'^2\, dr' \int_0^\pi sin\theta' d\theta' \int_0^{2\pi} d\phi'$

Now,

$$f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int_0^\infty r'^2 v(r')\ dr' \int_0^\pi e^{\iota q r' cos\theta'} sin\theta' d\theta' \int_0^{2\pi} d\phi'$$

After calculation we get,

$$f(\theta) = \frac{2\mu}{q\hbar^2}\int_0^\infty r' v(r') \sin qr'\ \ dr'$$

Therefore, differential cross section is,

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

2.3 Yukawa Potential

Yukawa potential is also called a screened Coulomb potential.

$$V_Y(r) = -g^2 \frac{e^{-mr\alpha}}{r}$$

where g is a magnitude scaling constant, i.e. is the amplitude of potential, m is the mass of the particle, r is the radial distance to the particle, and **α** is another scaling constant, so that $r \approx \frac{1}{\alpha m}$ is the approximate range. The potential is monotonically increasing in r and it is negative, implying the force is attractive. In the SI system, the unit of the Yukawa potential is (1/meters).

The coulomb potential of electromagnetism is an example of a Yukawa potential with the $e^{-\alpha m r}$ factor equal to 1 (here photon mass $m = 0$).

If the particle has no mass, then the Yukawa potential reduces to a Coulomb potential and the range is infinite.

When m = 0

$$Vy(r) = -g^2 \frac{e^0}{r}$$

$$Vy(r) = -g^2 \frac{1}{r}$$

It is called coulomb potential where $g^2 = \frac{q_1 q_2}{4\pi\epsilon_0}$

Yukawa potential is same as the Coulomb Potential except the sign and the exponential factor. The Coulomb potential has effect over a greater distance whereas the Yukawa potential approaches zero rather quickly. Yukawa potential or Coulomb potential is non-zero for any large r.

If we put the coulomb potential in born approximation, it will be divergence at a point. But for Yukawa potential it will be converging from a point that's why we get attractive force. The Born approximation [22] also tells us that, in a spherically symmetrical potential, we can approximate the outgoing scattered wave function as the sum of incoming plane wave function and a small perturbation.

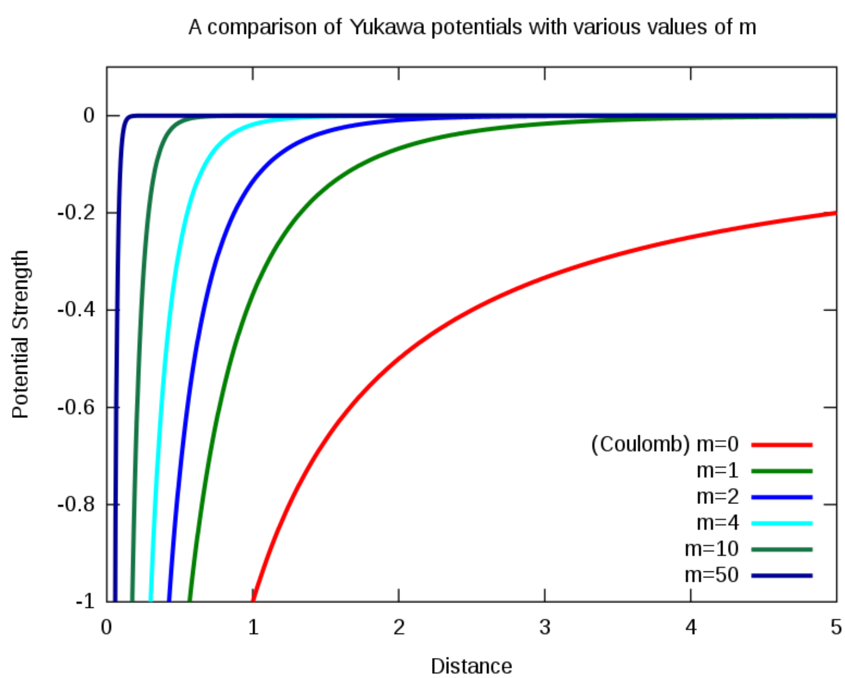

Fig 2.3: A comparison of Yukawa potential where $g = 1$ and with various values for m.

Yukawa Interaction:

Yukawa interaction is an interaction between particles according to the Yukawa potential. This is used to describe the nuclear force between nucleons mediated by pions. This potential has different sign from Coulomb potential and this sign make the Yukawa interaction attractive because Yukawa particle has spin zero and even spin always results in an attractive potential.

We can find nuclear force between nucleons by this type of potential. Also, we see that if we apply Born approximation in Yukawa potential it will be converging from a point and for this attractive force can be obtained. In our work we deal with soft core sphere scattering in which we can apply Born Approximation. So, we use Yukawa potential.

Scattering Cross Section (Coulomb Potential & Yukawa Potential):

<u>Using Coulomb Potential</u>

Considering the Coulomb Potential:

$$V_C(r) = -\frac{q_e^2}{r} \qquad [q_e \text{ is the charge of an electron}]$$

Spherically symmetric Born approximation for scattering:

$$f(\theta) = -\frac{2m}{\hbar^2 k}\int_0^\infty \sin(kr')\, V(r')r'dr'$$

Now plugin the potential into $f(\theta)$:

$$f(\theta) = \frac{2m}{\hbar^2 k}\int_0^\infty \sin(kr')\frac{q_e^2}{r'}r'dr'$$

$$\rightarrow f(\theta) = \frac{2mq_e^2}{\hbar^2 k}\int_0^\infty \sin(kr')\,dr'$$

Here we get integral of sin (kr) from $0\ to\ \infty$. Integral of sin (kr) from $0\ to\ \infty$ diverges.

In order to get a convergence the Yukawa potential is considered.

Using Yukawa Potential

Consider Yukawa potential:

$$V_Y(r) = -V_0\frac{e^{-\frac{r}{R}}}{r} \qquad ;\ [V_0 = g^2\ \ \&\ \ \alpha m = \tfrac{1}{R}]$$

Now plug in the Yukawa potential into $f(\theta)$:

$$f(\theta) = \frac{2m}{\hbar^2 k}\int_0^\infty \sin(kr')\,V_0\frac{e^{-\frac{r\prime}{R}}}{r}r'dr'$$

$$\rightarrow f(\theta) = \frac{2mV_0}{\hbar^2 k}\int_0^\infty \sin(kr')\,e^{-\frac{r\prime}{R}}dr'$$

Here,

$$\int e^{-ax}\sin(\mathrm{bx})dx = -\frac{be^{-ax}\cos(bx)}{b^2+a^2} - \frac{ae^{-ax}\sin(bx)}{b^2+a^2} + C$$

$$\therefore f(\theta) = \frac{2mV_0}{\hbar^2 k}\frac{k}{\frac{1}{R^2}+k^2}$$

$$\rightarrow f(\theta) = \frac{2mV_0}{\hbar^2}\frac{R^2}{1+(kR)^2}$$

We know that scattering cross section: $\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{4m^2V_0^2}{\hbar^4}\frac{R^4}{(1+(kR)^2)^2}$$

Now take limit $kR \rightarrow 0$ (Low energy):

$$\lim_{kR\rightarrow 0}(\frac{d\sigma}{d\Omega}) = \frac{4m^2V_0^2}{\hbar^4}\frac{R^4}{(1+(0)^2)^2}$$

$$\rightarrow \lim_{kR\rightarrow 0}(\frac{d\sigma}{d\Omega}) = \frac{4m^2V_0^2R^4}{\hbar^4}$$

From this we see that using Yukawa potential we get convergence for the scattering cross section. So we get attractive force.

Chapter 3: Stress Tensor & Derivation of Quantum Mechanical Force Equations

3.1 Tensor Background

As we all know electromagnetic wave carries momentum. So, when an electromagnetic collide with an object the momentum transfers to its colliding object. But we need to know the amount of momentum transferred to the object. For this we use Maxwell stress tensor.

We know, the force exerted by a electromagnetic wave for a charge cluster contained in volume 'v' can be expressed as:

$$F = \int_{v} \rho(E + v \times B)\, \partial\tau$$

$$F = \int_{v} \rho E + \rho(v \times B)\, \partial\tau$$

And we know $\rho v = J$

So the equation becomes

$$F = \int_{v} \rho E + (J \times B)\, \partial\tau$$

And we can write the force equation in a unit volume is

$$f = \int_{v} \rho E + (J \times B)$$

We can write this formula in terms of electric and magnetic field. From the Maxwell equations we know

$$\nabla . E = \frac{\rho}{\epsilon_0}$$

$$\rho = \epsilon_0 (\nabla . E)$$

and

$$\nabla \times B = J_c + J_D$$

$$\nabla \times B = \mu_0 + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$J = \frac{1}{\mu_0} (\nabla \times B) - \epsilon_0 \frac{\partial E}{\partial t}$$

So, our equation becomes

$$f = \epsilon_0 (\nabla . E) E + (\frac{1}{\mu_0} (\nabla \times B) - \epsilon_0 \frac{\partial E}{\partial t}) \times B$$

If we apply chain rule of differentiation on this $\frac{\partial}{\partial t} (E \times B)$

$$\frac{\partial}{\partial t} (E \times B) = E \times \frac{\partial E}{\partial t} + \frac{\partial E}{\partial t} \times B = \frac{\partial E}{\partial t} \times B + E \times \frac{\partial b}{\partial t}$$

From Maxwell Faraday's law

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\frac{\partial B}{\partial t} = -(\nabla \times E)$$

We can separate $\frac{\partial}{\partial t}(E \times B)$ from E$\times \frac{\partial B}{\partial t}$

$$\frac{\partial}{\partial t}(E \times B) = \frac{\partial E}{\partial t} \times B + E \times (-\nabla \times E)$$

$$\frac{\partial}{\partial t}(E \times B) = \frac{\partial E}{\partial t} \times B - E \times (\nabla \times E)$$

$$\frac{\partial}{\partial t} \times B = \frac{\partial E}{\partial t}(E \times B) + E \times (\nabla \times E)$$

So the equation for force in per unit volume will be

$$f = \epsilon_0 \left[(\nabla . E)E - \frac{1}{2}\nabla(E^2) + (E.\nabla)E\right] - \frac{1}{\mu_0}\left[(\nabla . B)B - \frac{1}{2}\nabla(B^2) + (B.\nabla)B\right] - \epsilon_0 \frac{\partial}{\partial t}(E \times B)$$

$$f = \epsilon_0[(\nabla . E)E + (E.\nabla)E] - \frac{1}{\mu_0}[(\nabla . B)B + (B.\nabla)B] - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 \frac{\partial}{\partial t}(E \times B)$$

We can simplify this equation using tensor. So the question arises what is tensor? In mathematics, a tensor is an algebraic object that describes a multilinear relationship between sets of algebraic objects related to a vector space. Objects that tensors may map between include vectors and scalars, and even other tensors. [TN1]

The maxwell's stress tensor is given below

$$\underset{T_{ij}}{\leftrightarrow} \equiv \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) - \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}\delta_{ij}B^2)$$

Here δ_{ij} is the dirac delta function and $(\delta_{xx} = \delta_{yy} = \delta_{zz} = 1),(\delta_{xy} = \delta_{xz} = \delta_{yx} = 0)$

From maxwell stress tensor we can prove all of them but here we are giving one from each

$$T_{xx} = \epsilon_0\left(E_x^2 - \frac{1}{2}\delta_{xx}\left(E_x^2 + E_y^2 + E_z^2\right) + \frac{1}{\mu_0}\left(B_x^2 - \frac{1}{2}\delta_{xx}\left(B_x^2 + B_y^2 + B_z^2\right)\right)\right)$$

$$T_{xx} = \epsilon_0(E_x^2 - \frac{1}{2}\left(E_x^2 + E_y^2 + E_z^2\right) + \frac{1}{\mu_0}\left(B_x^2 - \frac{1}{2}\left(B_x^2 + B_y^2 + B_z^2\right)\right) \qquad \text{AS } \delta_{xx} = 1$$

$$T_{xx} = \frac{1}{2}\epsilon_0\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right)$$

And

$$T_{xy} = \epsilon_0\left(E_x E_y - \frac{1}{2}\delta_{xy}\left(E_x^2 + E_y^2 + E_z^2\right) + \frac{1}{\mu_0}\left(B_x B_y - \frac{1}{2}\delta_{xx}\left(B_x^2 + B_y^2 + B_z^2\right)\right)\right)$$

$$T_{xy} = \epsilon_0 E_x E_y + \frac{1}{\mu_0}B_x B_y \qquad \text{as } \delta_{xy} = 0$$

In this equation the magnetic and electric field are the values of where we are trying to find the stress tensor of. The value of the stress tensor will depend on the electric field and will change with the electric field.

The scaler product of a vector field and tensor is given below

$$\left(\underset{a_i}{\rightarrow} . \underset{T_{ij}}{\leftrightarrow}\right)_j = \sum_{I=x,y,z} a_i T_{ij}$$

And the divergence of a tensor will be

$$\left(\nabla_i . \underset{T}{\leftrightarrow}\right)_j = \sum_{i=x,y,z} \nabla_i T_{ij}$$

$$= \sum \nabla_i [\epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

$$= \sum [\epsilon_0 \left((\nabla_i . E_i)E_j + (E_i . \nabla_i)E_i - \frac{1}{2}\delta_{ij}\nabla_i E^2\right) + \frac{1}{\mu_0}\left((\nabla_i . B_{i_i})B_j + (B_i . \nabla_i)B_i - \frac{1}{2}\delta_{ij}\nabla_i B^2\right)$$

$$= \sum [\epsilon_0 \left((\nabla_i . E_i)E_j + (E_i . \nabla_i)E_i - \frac{1}{2}\nabla_i E^2\right) + \frac{1}{\mu_0}\left((\nabla_i . B_{i_i})B_j + (B_i . \nabla_i)B_i - \frac{1}{2}\nabla_i B^2\right)$$ If i=j

Ithe above equation $\nabla_i . E_i$ is the divergence of E and $\nabla_i B^2$ is the gradient of B. so the force equation will be

$$\underset{f}{\rightarrow} = \nabla . \underset{T}{\leftrightarrow} - \epsilon_0 \mu_0 \frac{\delta}{\delta t} S$$

And the equation for total force will be

$$F = \int_v (\nabla.\underset{T}{\leftrightarrow})\delta\tau - \epsilon_0\mu_0 \int_v \frac{\delta}{\delta t} S\, \delta\tau$$

In this equation if we use Gauss divergence role the volume interigal will become surface integral

$$F = \int_v (\nabla.\underset{T}{\leftrightarrow})da - \epsilon_0\mu_0 \frac{\delta}{\delta t} \int_v S\, \delta\tau$$

Using this equation we can find the eletromagnetic force on the charge inside a volume. In this equation we use the surafce integal because during static position the 2nd term will be time independent. For this phenomenon the 2nd part will be terminted and we can fine the total force the charges using the stress tensor.

From newton;s 2nd law of motion we know When a body is acted upon by a force, the time rate of change of its momentum equals the force. Which means

$F = \frac{d}{dt} P_{mech}$ here P_{mech} is the mechanical momentum

If we want to apply this law foe electromagnetic wave, we will get

$$\frac{d}{dt} P_{mech} = -\epsilon_0\mu_0 \frac{d}{dt} \int_v S d\tau + \oint_S \underset{T}{\leftrightarrow} da$$

Here P_{mech} is the total momentum of the charges inside the volume. And differentiation signify the change of this momentum. The volume integral is the momentum of the electromagnetic field.

$$p_{em} = \mu_0 \epsilon_0 \int_V S\, d\tau$$

And the 2nd term is for the outgoing of momentum of the volume. Here the change of momentum is equal to the momentum of the electromagnetic field working on the volume 'v'.

$$\oint_S \underset{T}{\leftrightarrow} da = \epsilon_0 \mu_0 \frac{d}{dt} \int_V S d\tau + \frac{d}{dt} P_{mech}$$

Which means the mechanical momentum of the charges inside the volume V and the electromagnetic field's momentum is equal to outgoing of momentum through the surface area S. we can clearly see a relation between the mechanical momentum and momentum of the electromagnetic field. And the relation is inversely proportional. So the density of the momentum will be $g = \mu_0 \epsilon_0 = \epsilon(E \times B)$ and $\underset{T}{\leftrightarrow} da$ is the outgoing electromagnetic momentum through the surface area.

In this if we see no change in the mechanical momentum it means the clustered of charge is in the free space.

$$\int \frac{\delta g}{\delta t} = \oint \underset{T}{\leftrightarrow} . da = \int \nabla . \underset{T}{\leftrightarrow} d\tau$$

$$\frac{\delta g}{\delta t} = \nabla . \underset{T}{\leftrightarrow}$$

This is the equation f electromagnetic momentum continuity where g is the density of the momentum q is the charge $\underset{T}{\leftrightarrow}$ flux density of the momentum S is the work done by the density flux. Using this differential equation, we are trying to find the momentum of the local field and as we are using 'nabla' operator it is working on the point charge.

3.2 EM Stress Tensor & Force Equations

We have derived the quantum-mechanical force equations in analogy with this photonics force equations derivation:

$$\frac{1}{2} \oiint \langle \overline{\overline{T}} \rangle \, \overrightarrow{ds} = \langle \vec{F} \rangle + 0$$

$$\langle \overrightarrow{F_{Re}} \rangle = \frac{1}{2} \mathrm{Re} \oiint \langle \overline{\overline{T}} \rangle \overrightarrow{ds}$$

$$\oiint \langle \overline{\overline{T}} \rangle \, \overrightarrow{ds} = \iiint \vec{f}(t) dv + \iiint \frac{\delta G(t)}{\delta t} dv \ ; \text{Libniz theorem}$$

Gauss divergence theorem:

$$\iiint \overline{\nabla} \cdot \overline{\overline{T}}\, dv = \iiint \vec{f}(t) dv + \iiint \frac{\delta G(t)}{\delta t} dv$$

$$\overline{\nabla} \cdot \overline{\overline{T}} = \vec{f} + \frac{\delta Gt)}{\delta t}$$

$$\vec{f} \rightarrow \text{force density } (\frac{N}{m^3})$$

$$\overline{\nabla} \cdot \overline{\overline{T}} \rightarrow \text{wave momentum flux density} \rightarrow \text{stress}$$

$$\overline{\overline{T}} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$\overline{\nabla} \cdot \overline{\overline{T}} = \begin{bmatrix} \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \end{bmatrix} \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} = \begin{bmatrix} (\overline{\nabla} \cdot \overline{\overline{T}})_x & (\overline{\nabla} \cdot \overline{\overline{T}})_y & (\overline{\nabla} \cdot \overline{\overline{T}})_z \end{bmatrix}$$

1x3 3x3 => 1x3

$$(\overline{\nabla} \cdot \overline{\overline{T}})_x = \frac{\delta}{\delta x} T_{xx} + \frac{\delta}{\delta x} T_{xy} + \frac{\delta}{\delta x} T_{xz}$$

$$\langle\bar{\bar{T}}\rangle = DE^* + BH^* - \frac{1}{2}\bar{\bar{I}}(\mu_0 H^2 + \epsilon_0 E^2)$$

$$C = \mu_0 H^2 + \epsilon_0 E^2$$

$$= \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} [E_x^* \quad E_y^* \quad E_z^*] + \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} [H_x^* \quad H_y^* \quad H_z^*] - \begin{bmatrix} \hat{\imath}\hat{\imath} & 0 & 0 \\ 0 & \hat{\jmath}\hat{\jmath} & 0 \\ 0 & 0 & \hat{k}\hat{k} \end{bmatrix} \{C\}$$

$$= \begin{bmatrix} (D_x E_x^*)\hat{\imath}\hat{\imath} & (D_x E_y^*)\hat{\imath}\hat{\jmath} & (D_x E_z^*)\hat{\imath}\hat{k} \\ (D_y E_x^*)\hat{\jmath}\hat{\imath} & (D_y E_y^*)\hat{\jmath}\hat{\jmath} & (D_y E_z^*)\hat{\jmath}\hat{k} \\ (D_z E_x^*)\hat{k}\hat{\imath} & (D_y E_x^*)\hat{k}\hat{\jmath} & (D_z E_z^*)\hat{k}\hat{k} \end{bmatrix} + \begin{bmatrix} (B_x H_x^*)\hat{\imath}\hat{\imath} & (B_x H_y^*)\hat{\imath}\hat{\jmath} & (B_x H_z^*)\hat{\imath}\hat{k} \\ (B_y H_x^*)\hat{\jmath}\hat{\imath} & (B_y H_y^*)\hat{\jmath}\hat{\jmath} & (B_y H_z^*)\hat{\jmath}\hat{k} \\ (B_z H_x^*)\hat{k}\hat{\imath} & (B_y H_x^*)\hat{k}\hat{\jmath} & (B_z H_z^*)\hat{k}\hat{k} \end{bmatrix}$$

$$- \begin{bmatrix} \{C\}\hat{\imath}\hat{\imath} & 0 & 0 \\ 0 & \{C\}\hat{\jmath}\hat{\jmath} & 0 \\ 0 & 0 & \{C\}\hat{k}\hat{k} \end{bmatrix}$$

$$\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} [D_x E_x^* + B_x H_x^* - C]\hat{\imath}\hat{\imath} & [D_x E_y^* + B_x H_y^*]\hat{\imath}\hat{\jmath} & [D_x E_z^* + B_x H_z^*]\hat{\imath}\hat{k} \\ [D_y E_x^* + B_y H_x^*]\hat{\jmath}\hat{\imath} & [D_y E_y^* + B_y H_y^* - C]\hat{\jmath}\hat{\jmath} & [D_y E_z^* + B_y H_z^*]\hat{\jmath}\hat{k} \\ [D_z E_x^* + B_z H_x^*]\hat{k}\hat{\imath} & [D_y E_x^* + B_y H_x^*]\hat{k}\hat{\jmath} & [D_z E_z^* + B_z H_z^* - C]\hat{k}\hat{k} \end{bmatrix}$$

$$\bar{\bar{T}} \cdot d\bar{s} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

where,

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \Delta x \Delta y \\ \Delta y \Delta z \\ \Delta z \Delta x \end{bmatrix}$$

$$\overline{\overline{T}} \cdot d\bar{s} = \begin{bmatrix} T_{xx}n_x + T_{xy}n_y + T_{xz}n_z \\ T_{yx}n_x + T_{yy}n_y + T_{yz}n_z \\ T_{zx}n_x + T_{zy}n_y + T_{zz}n_z \end{bmatrix} = \begin{bmatrix} f_{x_{surf}} \\ f_{y_{surf}} \\ f_{z_{surf}} \end{bmatrix}$$

$$f_{x_{surf}} = \{D_x E_x^* + B_x H_x^* - C\}n_x + \left(D_x E_y^* + B_x H_y^*\right)n_y + (D_x E_z^* + B_x H_z^*)n_z$$

$$f_{y_{surf}} = \left(D_y E_x^* + B_y H_x^*\right)n_x + \{D_y E_y^* + B_y H_y^* - C\}n_y + \left(D_y E_z^* + B_y H_z^*\right)n_z$$

$$f_{y_{surf}} = (D_z E_x^* + B_z H_x^*)n_x + (D_y E_x^* + B_y H_x^*)n_y + \{D_z E_z^* + B_z H_z^* - C\}n_z$$

$$f_{x_{surf}} = \{C - D_x^* E_x - B_x^* H_x\}\mathbf{n_x} - \left(D_x^* E_y + B_x^* H_y\right)\mathbf{n_y} - (D_x^* E_z + B_x^* H_z)\mathbf{n_z}$$

$$f_{y_{surf}} = \left(-D_y^* E_x - B_y^* H_x\right)\mathbf{n_x} + \left\{C - D_y^* E_y - B_y^* H_y\right\}\mathbf{n_y} - \left(D_y^* E_z + B_y^* H_z\right)\mathbf{n_z}$$

$$f_{z_{surf}} = (-D_z^* E_x - B_z^* H_x)\mathbf{n_x} + \left(-D_z^* E_y - B_z^* H_y\right)\mathbf{n_y} + \{C - D_z^* E_z - B_z^* H_z\}\mathbf{n_z}$$

3.3 Quantum-Mechanical Stress Tensor & Force Equations

$$\overleftrightarrow{T} = \frac{\hbar}{2\mu}(\nabla\Psi \otimes \nabla\Psi^* + \nabla\Psi^* \otimes \nabla\Psi) + \overleftrightarrow{I}\left((E-U)|\Psi|^2 - \frac{\hbar^2}{\mu}|\nabla\Psi|^2\right)$$

Here,

$$\nabla\Psi = \begin{bmatrix} \frac{\delta}{\delta x} \\ \frac{\delta}{\delta y} \\ \frac{\delta}{\delta z} \end{bmatrix}\Psi \qquad \& \qquad \nabla\Psi^* = \begin{bmatrix} \frac{\delta}{\delta x} \\ \frac{\delta}{\delta y} \\ \frac{\delta}{\delta z} \end{bmatrix}\Psi^*$$

$$\text{Or}, \nabla\Psi = \begin{bmatrix} \frac{\delta\Psi}{\delta x} \\ \frac{\delta\Psi}{\delta y} \\ \frac{\delta\Psi}{\delta z} \end{bmatrix} \qquad \& \qquad \nabla\Psi^* = \begin{bmatrix} \frac{\delta\Psi^*}{\delta x} \\ \frac{\delta\Psi^*}{\delta y} \\ \frac{\delta\Psi^*}{\delta z} \end{bmatrix}$$

Tensor product between two vectors can be expressed as:

$$(\nabla\Psi) \otimes (\nabla\Psi^*) = (\nabla\Psi) \cdot (\nabla\Psi^*)^T$$

where,

$$(\nabla\Psi^*)^T = \begin{bmatrix} \frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi^*}{\delta z} \end{bmatrix}$$

$$(\nabla\Psi) \otimes (\nabla\Psi^*) = (\nabla\Psi\ ) \ \cdot (\nabla\Psi^*)^{T} = \begin{bmatrix} \frac{\delta\Psi}{\delta x} \\ \frac{\delta\Psi}{\delta y} \\ \frac{\delta\Psi}{\delta z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi^*}{\delta z} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\delta\Psi}{\delta x} \\ \frac{\delta\Psi}{\delta y} \\ \frac{\delta\Psi}{\delta z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi^*}{\delta z} \end{bmatrix} = \begin{bmatrix} \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z} \\ \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z} \\ \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z} \end{bmatrix}$$

again,

$$(\nabla\Psi^*) \otimes (\nabla\Psi) = (\nabla\Psi^*) \ \cdot (\nabla\Psi)^{T} = \begin{bmatrix} \frac{\delta\Psi^*}{\delta x} \\ \frac{\delta\Psi^*}{\delta y} \\ \frac{\delta\Psi^*}{\delta z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\delta\Psi}{\delta x} & \frac{\delta\Psi}{\delta y} & \frac{\delta\Psi}{\delta z} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\delta\Psi^*}{\delta x} \\ \frac{\delta\Psi^*}{\delta y} \\ \frac{\delta\Psi^*}{\delta z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\delta\Psi}{\delta x} & \frac{\delta\Psi}{\delta y} & \frac{\delta\Psi}{\delta z} \end{bmatrix} = \begin{bmatrix} \frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z} \\ \frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z} \\ \frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z} \end{bmatrix}$$

Now, we calculate the first parentheses of the Stress Tensor:

$$(\nabla\Psi \otimes \nabla\Psi^* + \nabla\Psi^* \otimes \nabla\Psi)$$

$$= \begin{bmatrix} \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z} \\ \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z} \\ \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x} & \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y} & \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z} \end{bmatrix} + \begin{bmatrix} \frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z} \\ \frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z} \\ \frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z} \\ \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z} \\ \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x} & \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y} & \frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z} \end{bmatrix}$$

This resulting tensor can be expressed as:

$$= \begin{bmatrix} \left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\imath}\hat{\imath} & \left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\imath}\hat{\jmath} & \left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\imath}\hat{k} \\ \left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\jmath}\hat{\imath} & \left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\jmath}\hat{\jmath} & \left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\jmath}\hat{k} \\ \left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{k}\hat{\imath} & \left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{k}\hat{\jmath} & \left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{k}\hat{k} \end{bmatrix}$$

We multiply $\frac{\hbar}{2\mu}$ with this tensor to calculate the first term of the Stress Tensor:

$$\overleftrightarrow{T_1} = \frac{\hbar}{2\mu}\begin{bmatrix} \left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\imath}\hat{\imath} & \left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\imath}\hat{\jmath} & \left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\imath}\hat{k} \\ \left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\jmath}\hat{\imath} & \left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\jmath}\hat{\jmath} & \left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\jmath}\hat{k} \\ \left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{k}\hat{\imath} & \left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{k}\hat{\jmath} & \left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{k}\hat{k} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\imath}\hat{\imath} & \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\imath}\hat{\jmath} & \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\imath}\hat{k} \\ \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\jmath}\hat{\imath} & \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\jmath}\hat{\jmath} & \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\jmath}\hat{k} \\ \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{k}\hat{\imath} & \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{k}\hat{\jmath} & \left(\frac{\hbar}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{k}\hat{k} \end{bmatrix}$$

Now we can proceed to the second term of the Stress Tensor:

$$\overleftrightarrow{T_2} = \overleftrightarrow{I}\left((E-U)|\Psi|^2 - \frac{\hbar^2}{\mu}|\nabla\Psi|^2\right)$$

$$= \begin{bmatrix} \hat{\imath}\hat{\imath} & 0 & 0 \\ 0 & \hat{\jmath}\hat{\jmath} & 0 \\ 0 & 0 & \hat{k}\hat{k} \end{bmatrix} \left((E - U)|\Psi|^2 - \frac{\hbar^2}{\mu} |\nabla\Psi|^2 \right)$$

$$= \begin{bmatrix} \left((E - U)|\Psi|^2 - \frac{\hbar^2}{\mu} |\nabla\Psi|^2 \right) \hat{\imath}\hat{\imath} & 0 & 0 \\ 0 & \left((E - U)|\Psi|^2 - \frac{\hbar^2}{\mu} |\nabla\Psi|^2 \right) \hat{\jmath}\hat{\jmath} & 0 \\ 0 & 0 & \left((E - U)|\Psi|^2 - \frac{\hbar^2}{\mu} |\nabla\Psi|^2 \right) \hat{k}\hat{k} \end{bmatrix}$$

therefore,

$$\overleftrightarrow{T} = \overleftrightarrow{T}_1 + \overleftrightarrow{T}_2$$

$$\overleftrightarrow{T}$$

$$= \begin{bmatrix} \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x} \cdot \frac{\delta\Psi^*}{\delta x} + \frac{\delta\Psi^*}{\delta x} \cdot \frac{\delta\Psi}{\delta x}\right)\hat{\imath}\hat{\imath} & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x} \cdot \frac{\delta\Psi^*}{\delta y} + \frac{\delta\Psi^*}{\delta x} \cdot \frac{\delta\Psi}{\delta y}\right)\hat{\imath}\hat{\jmath} & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x} \cdot \frac{\delta\Psi^*}{\delta z} + \frac{\delta\Psi^*}{\delta x} \cdot \frac{\delta\Psi}{\delta z}\right)\hat{\imath}\hat{k} \\ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y} \cdot \frac{\delta\Psi^*}{\delta x} + \frac{\delta\Psi^*}{\delta y} \cdot \frac{\delta\Psi}{\delta x}\right)\hat{\jmath}\hat{\imath} & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y} \cdot \frac{\delta\Psi^*}{\delta y} + \frac{\delta\Psi^*}{\delta y} \cdot \frac{\delta\Psi}{\delta y}\right)\hat{\jmath}\hat{\jmath} & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y} \cdot \frac{\delta\Psi^*}{\delta z} + \frac{\delta\Psi^*}{\delta y} \cdot \frac{\delta\Psi}{\delta z}\right)\hat{\jmath}\hat{k} \\ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z} \cdot \frac{\delta\Psi^*}{\delta x} + \frac{\delta\Psi^*}{\delta z} \cdot \frac{\delta\Psi}{\delta x}\right)\hat{k}\hat{\imath} & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z} \cdot \frac{\delta\Psi^*}{\delta y} + \frac{\delta\Psi^*}{\delta z} \cdot \frac{\delta\Psi}{\delta y}\right)\hat{k}\hat{\jmath} & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z} \cdot \frac{\delta\Psi^*}{\delta z} + \frac{\delta\Psi^*}{\delta z} \cdot \frac{\delta\Psi}{\delta z}\right)\hat{k}\hat{k} \end{bmatrix}$$

$$+ \begin{bmatrix} \left((E - U)|\Psi|^2 - \frac{\hbar^2}{2\mu} |\nabla\Psi|^2 \right) \hat{\imath}\hat{\imath} & 0 & 0 \\ 0 & \left((E - U)|\Psi|^2 - \frac{\hbar^2}{2\mu} |\nabla\Psi|^2 \right) \hat{\jmath}\hat{\jmath} & 0 \\ 0 & 0 & \left((E - U)|\Psi|^2 - \frac{\hbar^2}{2\mu} |\nabla\Psi|^2 \right) \hat{k}\hat{k} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\imath}\hat{\imath}+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\hat{\imath}\hat{\imath} & 0 & 0 \\ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\jmath}\hat{\imath} & 0 & 0 \\ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{k}\hat{\imath} & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\imath}\hat{\jmath} & 0 \\ 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\jmath}\hat{\jmath}+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\hat{\jmath}\hat{\jmath} & 0 \\ 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{k}\hat{\jmath} & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\imath}\hat{k} \\ 0 & 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\jmath}\hat{k} \\ 0 & 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{k}\hat{k}+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\hat{k}\hat{k} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x}\right)+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right]\hat{\imath}\hat{\imath} & 0 & 0 \\ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\jmath}\hat{\imath} & 0 & 0 \\ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{k}\hat{\imath} & 0 & 0 \end{bmatrix}$$

$$
+\begin{bmatrix} 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\imath}\hat{\jmath} & 0 \\ 0 & \left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y}\right)+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right]\hat{\jmath}\hat{\jmath} & 0 \\ 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{k}\hat{\jmath} & 0 \end{bmatrix}
$$

$$
+\begin{bmatrix} 0 & 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\imath}\hat{k} \\ 0 & 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\jmath}\hat{k} \\ 0 & 0 & \left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z}\right)+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right]\hat{k}\hat{k} \end{bmatrix}
$$

$$\overleftrightarrow{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta x}\right)+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right]\hat{\imath}\hat{\imath} & 0 & 0 \\ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{\jmath}\hat{\imath} & 0 & 0 \\ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x}\right)\hat{k}\hat{\imath} & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{\imath}\hat{\jmath} & 0 \\ 0 & \left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta y}\right)+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right]\hat{\jmath}\hat{\jmath} & 0 \\ 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y}\right)\hat{k}\hat{\jmath} & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta x}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\imath}\hat{k} \\ 0 & 0 & \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta y}\cdot\frac{\delta\Psi}{\delta z}\right)\hat{\jmath}\hat{k} \\ 0 & 0 & \left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z}\right)+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right]\hat{k}\hat{k} \end{bmatrix}$$

$$\overleftrightarrow{T}\cdot d\bar{s} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}\cdot\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

where,

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \Delta x \Delta y \\ \Delta y \Delta z \\ \Delta z \Delta x \end{bmatrix}$$

$$\overleftrightarrow{T} \cdot d\bar{s} = \begin{bmatrix} T_{xx} n_x + T_{xy} n_y + T_{xz} n_z \\ T_{yx} n_x + T_{yy} n_y + T_{yz} n_z \\ T_{zx} n_x + T_{zy} n_y + T_{zz} n_z \end{bmatrix}$$

$$= \begin{bmatrix} f_{x_{surf}} \\ f_{y_{surf}} \\ f_{z_{surf}} \end{bmatrix}$$

$$f_{x_{surf}} = \left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x} \cdot \frac{\delta\Psi^*}{\delta x} + \frac{\delta\Psi^*}{\delta x} \cdot \frac{\delta\Psi}{\delta x}\right) + \left((E-U)|\Psi|^2 - \frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right] n_x$$

$$+ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x} \cdot \frac{\delta\Psi^*}{\delta y} + \frac{\delta\Psi^*}{\delta x} \cdot \frac{\delta\Psi}{\delta y}\right) n_y + \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta x} \cdot \frac{\delta\Psi^*}{\delta z} + \frac{\delta\Psi^*}{\delta x} \cdot \frac{\delta\Psi}{\delta z}\right) n_z$$

$$f_{y_{surf}} = \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y} \cdot \frac{\delta\Psi^*}{\delta x} + \frac{\delta\Psi^*}{\delta y} \cdot \frac{\delta\Psi}{\delta x}\right) n_x$$

$$+ \left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y} \cdot \frac{\delta\Psi^*}{\delta y} + \frac{\delta\Psi^*}{\delta y} \cdot \frac{\delta\Psi}{\delta y}\right) + \left((E-U)|\Psi|^2 - \frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right] n_y$$

$$+ \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta y} \cdot \frac{\delta\Psi^*}{\delta z} + \frac{\delta\Psi^*}{\delta y} \cdot \frac{\delta\Psi}{\delta z}\right) n_z$$

$$f_{z_{surf}} = \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta x}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta x}\right)n_x + \left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta y}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta y}\right)n_y$$

$$+\left[\left(\frac{\hbar^2}{2\mu}\right)\left(\frac{\delta\Psi}{\delta z}\cdot\frac{\delta\Psi^*}{\delta z}+\frac{\delta\Psi^*}{\delta z}\cdot\frac{\delta\Psi}{\delta z}\right)+\left((E-U)|\Psi|^2-\frac{\hbar^2}{2\mu}|\nabla\Psi|^2\right)\right]n_z$$

Chapter 4: COMSOL 6.0 Implementation of the Experiment

4.1 Implementation of QM Force Equations in COMSOL

Several versions of the derived force equations have been implemented in COMSOL – which are still subject to further scrutiny and modifications. The versions of the equations are given below:

Table 4.1: Version 1: Equations with schr.Pr

fx_surf	**(**c1*(pd(schr.Psi,x)*pd(conj(schr.Psi),x)+pd(conj(schr.Psi),x)*pd(schr.Psi,x)**+(c2)**)*nx +c1*(pd(schr.Psi,x)*pd(conj(schr.Psi),y)+pd(conj(schr.Psi),x)*pd(schr.Psi,y))*ny+c1*(pd(schr.Psi,x)*pd(conj(schr.Psi),z)+pd(conj(schr.Psi),x)*pd(schr.Psi,z))*nz**)*schr.Pr**

fy_surf	c1*(pd(schr.Psi,y)*pd(conj(schr.Psi),z)+pd(conj(schr.Psi),y)*pd(schr.Psi,z))*nx+c1*(pd(schr.Psi,y)*pd(conj(schr.Psi),y)+pd(conj(schr.Psi),y)*pd(schr.Psi,y)**+(c2)**)*ny+c1*(pd(schr.Psi,y)*pd(conj(schr.Psi),z)+pd(conj(schr.Psi),y)*pd(schr.Psi,z))*nz**)*schr.Pr**
fz_surf	**(**c1*(pd(schr.Psi,z)*pd(conj(schr.Psi),x)+pd(conj(schr.Psi),z)*pd(schr.Psi,x))*nx+c1*(pd(schr.Psi,z)*pd(conj(schr.Psi),y)+pd(conj(schr.Psi),z)*pd(schr.Psi,y))*ny+c1*(pd(s chr.Psi,z)*pd(conj(schr.Psi),z)+pd(conj(schr.Psi),z)*pd(schr.Psi,z)+**(c2)**)*nz**)*schr.Pr**

Table 4.2: Version 2: ODE

fx_surf	**tdx1**= d(schr.Psi,x)*d(conj(schr.Psi),x)+d(conj(schr.Psi),x)*d(schr.Psi,x) **tdx2**= d(schr.Psi,x)*d(conj(schr.Psi),y)+d(conj(schr.Psi),x)*d(schr.Psi,y) **tdx3**= d(schr.Psi,x)*d(conj(schr.Psi),z)+d(conj(schr.Psi),x)*d(schr.Psi,z) **c1 =** (hbar^2)/(2*m_He) **c2 = (E-U)*(schr.Psi*conj(schr.Psi))-c1*(abs(gradient(schr.Psi)))^2*1[m^-2]** fx_dr = (**c1*tdx1+c2**)*nx+(**c1*tdx2**)*ny+(**c1*tdx3**)*nz
fy_surf	**tdy1**= d(schr.Psi,y)*pd(conj(schr.Psi),x)+d(conj(schr.Psi),y)*d(schr.Psi,x) **tdy2**= d(schr.Psi,y)*pd(conj(schr.Psi),y)+d(conj(schr.Psi),y)*d(schr.Psi,y) **tdy3**= d(schr.Psi,y)*pd(conj(schr.Psi),z)+d(conj(schr.Psi),y)*d(schr.Psi,z) fy_dr = (**c1*tdy1**)*nx+(**c1*tdy2+c2**)*ny+(**c1*tdy3**)*nz
fz_surf	**tdz1**= d(schr.Psi,z)*pd(conj(schr.Psi),x)+d(conj(schr.Psi),z)*d(schr.Psi,x) **tdz2**= d(schr.Psi,z)*pd(conj(schr.Psi),y)+d(conj(schr.Psi),z)*d(schr.Psi,y) **tdz3**= d(schr.Psi,z)*pd(conj(schr.Psi),z)+d(conj(schr.Psi),z)*d(schr.Psi,z)

	fz_dr = (**c1*tdz1**)*nx+(**c1*tdz2**)*ny+(**c1*tdz3+c2**)*nz

Table 4.3: Version 3: Final PDE

tdx1 = pd(psi,x)*pd(conj(psi),x)+pd(conj(psi),x)*pd(psi,x) $1/m^2$ tdx2 = pd(psi,x)*pd(conj(psi),y)+pd(conj(psi),x)*pd(psi,y) $1/m^2$ tdx3 = pd(psi,x)*pd(conj(psi),z)+pd(conj(psi),x)*pd(psi,z) $1/m^2$ tdy1 = pd(psi,y)*pd(conj(psi),x)+pd(conj(psi),y)*pd(psi,x) $1/m^2$ tdy2 = pd(psi,y)*pd(conj(psi),y)+pd(conj(psi),y)*pd(psi,y) $1/m^2$ tdy3 = pd(psi,y)*pd(conj(psi),z)+pd(conj(psi),y)*pd(psi,z) $1/m^2$ tdz1 = pd(psi,z)*pd(conj(psi),x)+pd(conj(psi),z)*pd(psi,x) $1/m^2$ tdz2 = pd(psi,z)*pd(conj(psi),y)+pd(conj(psi),z)*pd(psi,y) $1/m^2$ tdz3 = pd(psi,z)*pd(conj(psi),z)+pd(conj(psi),z)*pd(psi,z) $1/m^2$ c2 = (psi*conj(psi)*(E-U0))-c1*(abs((pd(psi,x)*nx+pd(psi,y)*ny+pd(psi,z)*nz)))^2 J
fx_dr = ((c1*tdx1+c2)*nx+(c1*tdx2)*ny+(c1*tdx3)*nz)*schr.Pr J/m^2 fy_dr = ((c1*tdy1)*nx+(c1*tdy2+c2)*ny+(c1*tdy3)*nz)*schr.Pr J/m^2 fz_dr = ((c1*tdz1)*nx+(c1*tdz2)*ny+(c1*tdz3+c2)*nz)*schr.Pr J/m^2

4.2 Parameters and Final Variables

Table 4.4: Parameters

Name	Expression	Value
F_N_analytical	(4*pi*(hbar^2)*A0^2)/m_He	Analytical F_N
c1	(hbar^2)/(2*m_He)	Coefficient 1
c2	0	
F_N_pull	-0.068*F_N	matter-wave pulling force for beam at alpha = 70 deg
qm_pb	1	permeability
E0	1.87*10^-3[eV]	energy scale
n_He	10^20[m^-3]	volume density of particles (rarefied beam)
m_heavy	100[amu]	heavy atom effective mass
E	13*E0	energy of the incident particle
F_N	2.1*10^-21 [N]	scale of quantum force
E_given	2.5*10^-2[eV]	E given at 300 K
qm_pv	3.9	permittivity
U0	-37.7*E0	potential energy
m_He	4[amu]	He effective mass
a0	5.29177210903*10^-11 [m]	Bohr radius
alpha	50	alpha > 39

hbar	h_const/2*pi	reduced Planck's constant
E_exact	k_B_const*300	E exact at 300 K
A0	sqrt(10^10)	beam amplitude
qm_ri	sqrt(1-U0/E)	quantum mechanical refractive index
k0	sqrt(2*m_He*E)/hbar	k naught
op_ri	sqrt(qm_pv*qm_pb)	optical refractive index from permeability & permittivity

Table 4.5: Variables

Name	**Expression**
tdx1	pd(psi,x)*pd(conj(psi),x)+pd(conj(psi),x)*pd(psi,x)
tdx2	pd(psi,x)*pd(conj(psi),y)+pd(conj(psi),x)*pd(psi,y)
tdx3	pd(psi,x)*pd(conj(psi),z)+pd(conj(psi),x)*pd(psi,z)
tdy1	pd(psi,y)*pd(conj(psi),x)+pd(conj(psi),y)*pd(psi,x)
tdy2	pd(psi,y)*pd(conj(psi),y)+pd(conj(psi),y)*pd(psi,y)
tdy3	pd(psi,y)*pd(conj(psi),z)+pd(conj(psi),y)*pd(psi,z)
tdz1	pd(psi,z)*pd(conj(psi),x)+pd(conj(psi),z)*pd(psi,x)
tdz2	pd(psi,z)*pd(conj(psi),y)+pd(conj(psi),z)*pd(psi,y)
tdz3	pd(psi,z)*pd(conj(psi),z)+pd(conj(psi),z)*pd(psi,z)
c2	(psi*conj(psi)*(E-U0))-c1*(abs((pd(psi,x)*nx+pd(psi,y)*ny+pd(psi,z)*nz)))^2
fx_dr	((c1*tdx1+c2)*nx+(c1*tdx2)*ny+(c1*tdx3)*nz)*schr.Pr
fy_dr	((c1*tdy1)*nx+(c1*tdy2+c2)*ny+(c1*tdy3)*nz)*schr.Pr

fz_dr	((c1*tdz1)*nx+(c1*tdz2)*ny+(c1*tdz3+c2)*nz)*schr.Pr
nz	0

4.3 Simulation Setup and Geometry

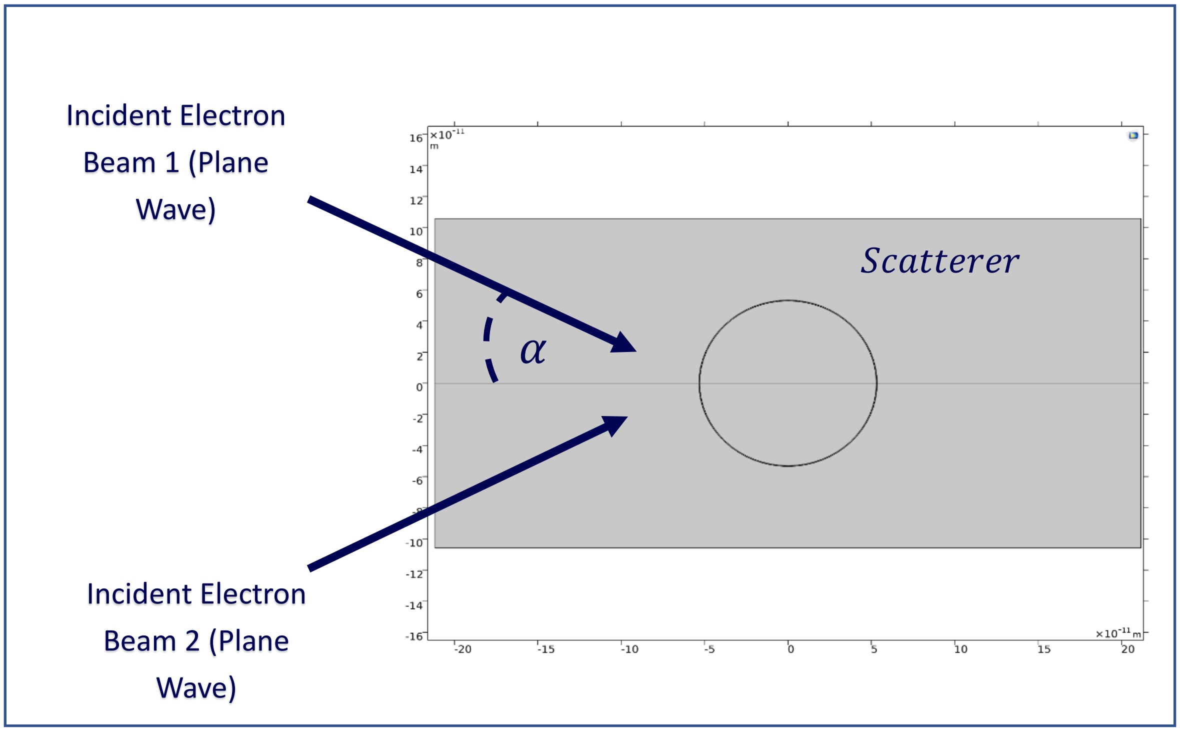

Figure 4.1: Experimental setup used in 2D

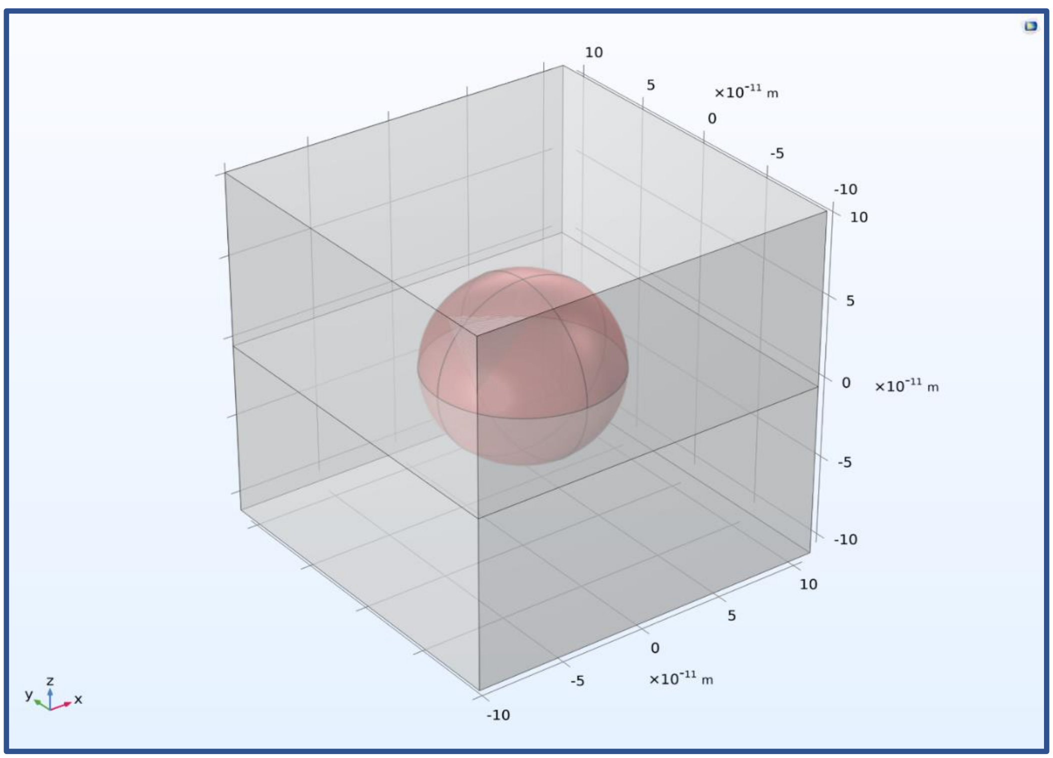

Figure 4.2: Model Geometry

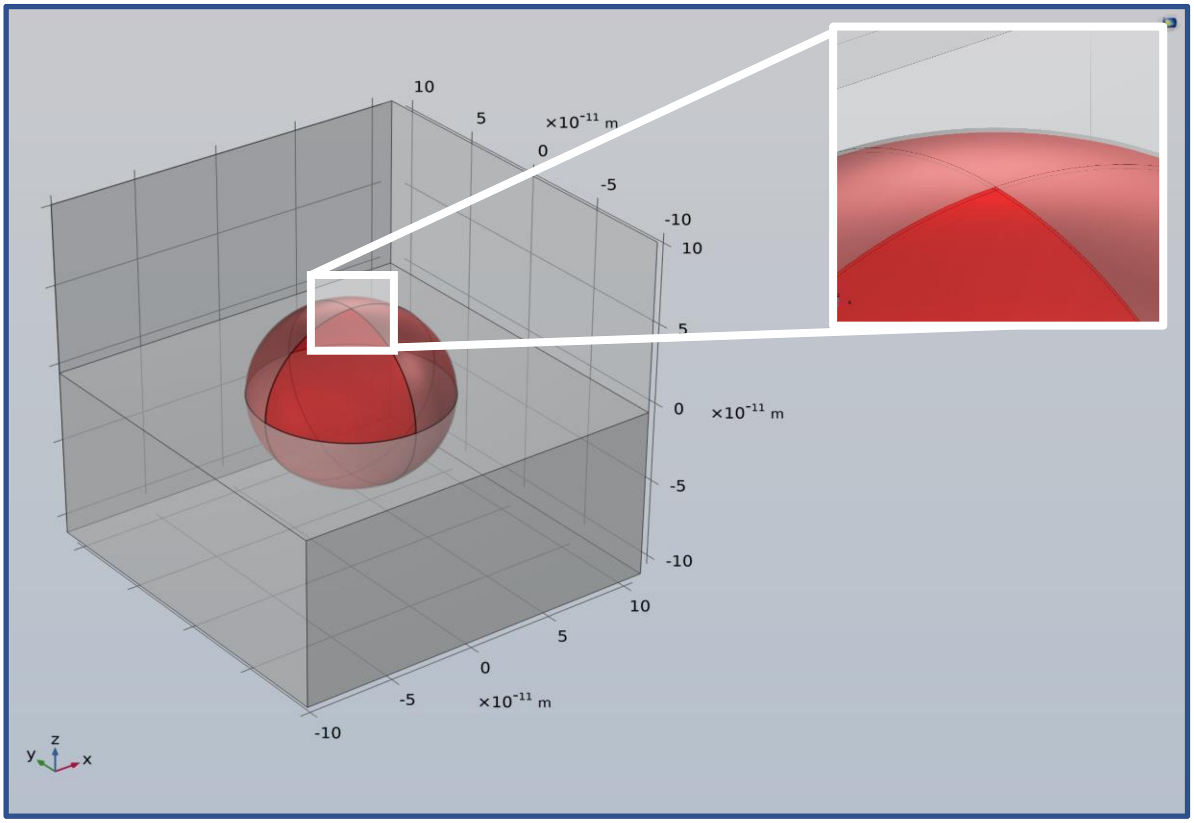

Figure 4.3: Model geometry breakdown: scatterer (red) enclosed by a sphere (transparent-white) with radius 1.01X the target particle.

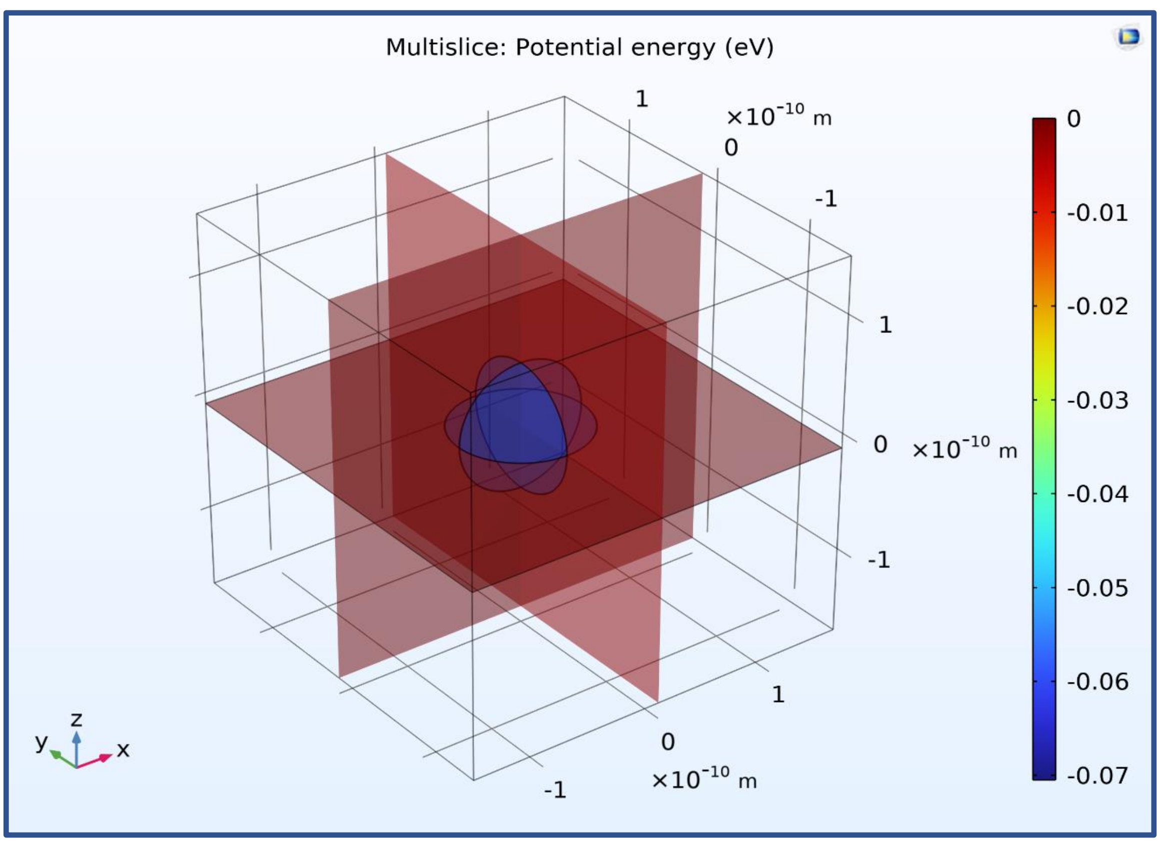

Figure 4.4: Electron Potential Energy (eV)

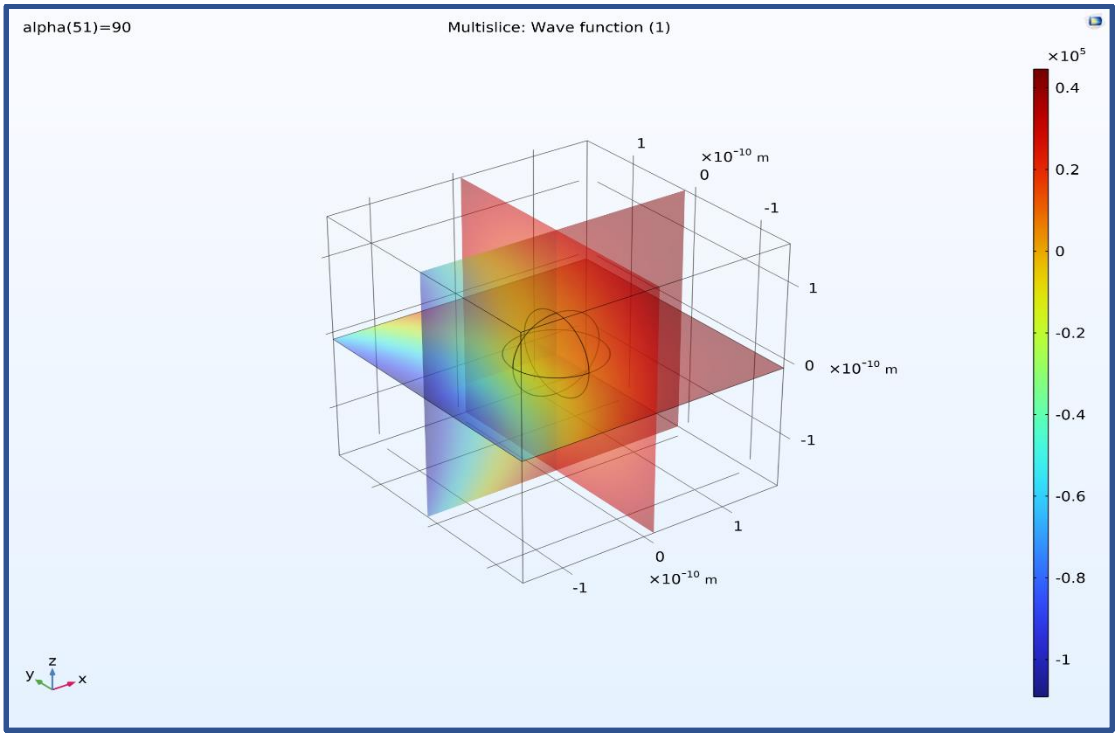

Figure 4.5: Wave function (real) of the quantum-mechanical system

4.4 Results & Analysis

Table 4.6 Results

alpha	fx_dr (J/m)	fy_dr (J/m)	fz_dr (J/m)
40	-3.60E-21	3.31E-28	7.10E-29
41	-6.53E-22	6.02E-29	1.08E-29
42	-3.84E-23	3.54E-30	9.06E-31
43	-1.80E-25	1.71E-32	9.57E-33
44	-2.08E-39	2.22E-46	2.48E-46
45	-2.08E-25	2.57E-32	4.43E-32
46	-3.47E-23	4.83E-30	1.05E-29
47	-5.35E-22	7.83E-29	1.84E-28
48	-2.94E-21	4.17E-28	9.32E-28
49	-8.36E-21	1.07E-27	1.99E-27
50	-1.36E-20	1.51E-27	1.92E-27
51	-1.28E-20	1.24E-27	8.45E-28
52	-6.78E-21	6.25E-28	1.90E-28
53	-1.90E-21	1.75E-28	3.25E-29

54	-2.29E-22	2.11E-29	4.11E-30
55	-6.10E-24	5.64E-31	1.98E-31
56	-1.78E-27	1.76E-34	1.37E-34
57	-3.53E-28	4.01E-35	5.58E-35
58	-3.10E-24	4.07E-31	7.90E-31
59	-1.35E-22	1.94E-29	4.42E-29
60	-1.23E-21	1.80E-28	4.19E-28
61	-4.96E-21	6.80E-28	1.43E-27
62	-1.11E-20	1.33E-27	2.16E-27
63	-1.42E-20	1.47E-27	1.48E-27
64	-1.04E-20	9.82E-28	4.72E-28
65	-4.25E-21	3.91E-28	8.97E-29
66	-8.62E-22	7.96E-29	1.41E-29
67	-6.15E-23	5.66E-30	1.34E-30
68	-4.85E-25	4.58E-32	2.30E-32
69	-6.67E-33	6.97E-40	7.23E-40
70	-7.04E-26	8.54E-33	1.41E-32
71	-2.13E-23	2.93E-30	6.20E-30
72	-4.00E-22	5.85E-29	1.37E-28
73	-2.44E-21	3.50E-28	7.93E-28

74	-7.52E-21	9.80E-28	1.88E-27
75	-1.32E-20	1.49E-27	2.02E-27
76	-1.33E-20	1.31E-27	9.85E-28
77	-7.63E-21	7.06E-28	2.38E-28
78	-2.35E-21	2.16E-28	4.14E-29
79	-3.23E-22	2.98E-29	5.59E-30
80	-1.13E-23	1.05E-30	3.32E-31
81	-9.99E-27	9.75E-34	6.90E-34
82	-2.22E-29	2.47E-36	3.23E-36
83	-1.51E-24	1.94E-31	3.65E-31
84	-9.21E-23	1.31E-29	2.95E-29
85	-9.70E-22	1.42E-28	3.32E-28
86	-4.28E-21	5.93E-28	1.27E-27
87	-1.03E-20	1.26E-27	2.14E-27
88	-1.42E-20	1.50E-27	1.63E-27
89	-1.12E-20	1.07E-27	5.71E-28
90	-4.97E-21	4.57E-28	1.13E-28

4.5 Observations & Comments:

Force of gravity for He particle is calculated to be $F_g = m_{He}g = 6.5093 \times 10^{-20}N$, which implies that $\frac{F}{F_g} \propto -10^2$, a pulling force exceeding the gravitation pull by two orders of magnitude.

We observe a consistency in the force values being pulling in nature in accordance with the existing literature; however, the values vary significantly with variation in incident angle: from 10^-20 to 10^-33 – which is not in accordance with the literature. The work we have presented here is not concluded and is still in examination and will remain so until we resolve all the discrepancies.

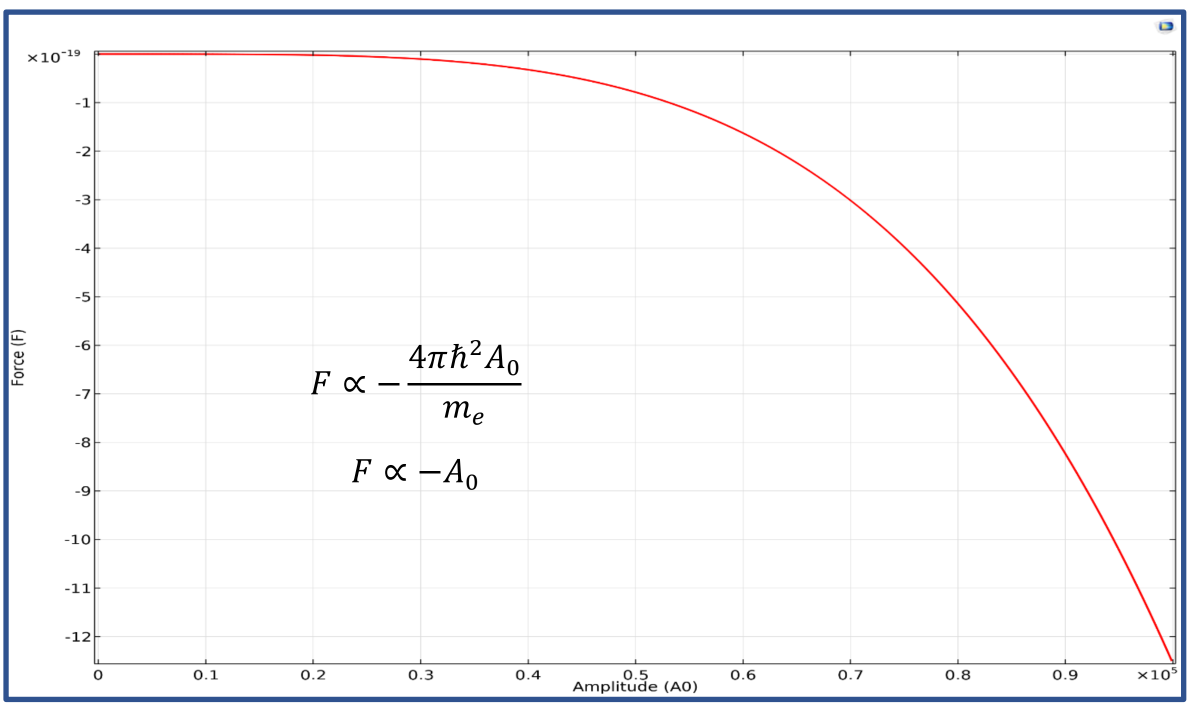

Figure 4.6: Parametric sweep of beam amplitude from 1 to A0 = 1E5 at $\alpha = 50\,°$, the correct mathematical relationship of the force being negatively proportional (pulling force) to the beam amplitude is observed.

Chapter 5: Conclusion

The matter-wave tractor beam we have implemented and intend to study further– has the potential to pull atoms into a microscopic hole. According to researchers, every atom that enters the tractor beam is pulled into the nanofibre – there is no escape. The atoms can be held for long periods once inside the beam. This opens up tremendous possibilities in enabling faster and more secure communication, which is regarded as quantum communication. It would be of high interest in the area of defence and security.

Furthermore, now with the development of quantum computers, storage based on quantum mechanical technology will also be an integral part of the complete development process. Such beams can also aid in this. Moreover, this is a very new concept and is yet to be used in biomedical science. In similar ways, we believe the components of blood can be moved and manipulated, shielding it from the disruptive effect of intense light simultaneously.

In recent times, efficient detection of disease has been a matter of great concern in medical science. A complete blood test can detect diseases such as cancer, malaria, anaemia, bacterial infection etc. however requires quite a few chemical processes and resources. Using a quantum tractor beam to detect such anomalies (at an atomic scale) would ameliorate the disease detection issues. A vital prospect of this project is that we get to manipulate objects down to the atomic scale, allowing us to detect biological aberrations and anomalies in cells at the earliest possible stage - broadening the boundaries of disease diagnosis and quantum biophysics.

5.1 Future Works

We have already implemented electron matter-wave tractor beams in COMSOL Multiphysics 6.0 and studied its interaction with a scatterer (Helium atom). Now, we are to further our study to interaction of the tractor beam under various conditions, such as a nanowaveguide mode, thereby enabling us to get a further insight into pulling and manipulation of objects using such beams. This would open up a wide array of possibilities for manipulation of objects within the realm of quantum mechanics (e.g., study of particle transport and binding, atom interferometry, quantum communication, quantum memory etc.) as it has several notable features, which have been unavailable in case of a photonic tractor beam.

An immediate & important area of study we are currently probing is the behavior of the tractor beam on a scatterer in semiconductor heterostructures. To model a heterostructure we use the effective mass approximation. It is common for electrons or holes to have different effective masses in different regions of a heterostructure. This can be easily specified by using multiple Effective Mass nodes in COMSOL. This an important aspect of study of interaction between non-biological objects, leading to noble findings.

Contrary to photonic tractor beams, the incident beam is composed of particles (e.g., electrons) and the forces exerted are of quantum probabilistic nature. This enables manipulation down to an atomic scale. Moreover, quantum tractor beams have the ability to attract scatterers with long range potential, which had always been a severe drawback of photonic tractor beams. Furthermore, our work will be extended to studying how such beams interact with biological cells/objects (similar to the ones used in the first part of our work) and this would allow extension of our work to manipulation of such cells, opening up new boundaries for detection of diseases (by finding particular molecules in a given sample).

References

[1] Ding, W., Zhu, T., Zhou, L.-M., & Qiu, C.-W. (2019). Photonic Tractor Beams: A Review. SPIE and CLP.

[2] Paul, N. K. (2016). Optical Pulling Force and Tractor Beam. *ProQuest*.

[3] S.-H. Lee, Y. Roichman, and D. G. Grier, “Optical solenoid beams,” Opt. Express 18(7), 6988–6993 (2010).

[4] Y. Roichman et al., “Optical forces arising from phase gradients,” Phys. Rev. Lett. 100(1), 013602 (2008).

[5] J. Chen et al., “Optical pulling force,” Nat. Photonics 5, 531–534 (2011)

[6] A. Novitsky, C.-W. Qiu, and H. Wang, “Single gradient less light beam drags particles as tractor beams,” Phys. Rev. Lett. 107(20), 203601 (2011)

[7] A. Mizrahi and Y. Fainman, “Negative radiation pressure on gain medium structures,” Opt. Lett. 35(20), 3405–3407 (2010).

[8] F. G. Mitri, “Optical Bessel tractor beam on active dielectric Rayleigh prolate and oblate spheroids,” J. Opt. Soc. Am. B 34(5), 899–908 (2017).

[9] X. Bian, D. L. Gao, and L. Gao, “Tailoring optical pulling force on gain coated nanoparticles with nonlocal effective medium theory,” Opt. Express 25(20), 24566–24578 (2017).

[10] K. Ding et al., “Realization of optical pulling forces using chirality,” Phys. Rev. A 89(6), 063825 (2014).

[11] D. E. Fernandes and M. G. Silveirinha, “Optical tractor beam with chiral light,” Phys. Rev. A 91(6), 061801 (2015).

[12] G. Li et al., “Wave propagation and Lorentz force density in gain chiral structures,” Opt. Mater. Express 6(2), 388–395 (2016).

[13] A. Novitsky and C.-W. Qiu, “Pulling extremely anisotropic lossy particles using light without intensity gradient,” Phys. Rev. A 90(5), 053815 (2014).

[14] S. De Liberato, “Light-matter decoupling in the deep strong coupling regime: the breakdown of the purcell effect,” Phys. Rev. Lett. 112(1), 016401 (2014)

[15] V. Kajorndejnukul et al., “Linear momentum increase and negative optical forces at dielectric interface,” Nat. Photonics 7(10), 787–790 (2013).

[16] M. I. Petrov et al., “Surface plasmon polariton assisted optical pulling force,” Laser Photonics Rev. 10(1), 116–122 (2016).

[17] V. Intaraprasonk and S. Fan, “Optical pulling force and conveyor belt effect in resonator-waveguide system,” Opt. Lett. 38(17), 3264–3267 (2013).

[18] M. Lewittes, S. Arnold, and G. Oster, “Radiometric levitation of micron sized spheres,” Appl. Phys. Lett. 40(6), 455–457 (1982).

[19] G. Wurm and O. Krauss, “Dust eruptions by photophoresis and solid state greenhouse effects,” Phys. Rev. Lett. 96(13), 134301 (2006).

[20] V. G. Shvedov et al., “Giant optical manipulation,” Phys. Rev. Lett. 105(11), 118103 (2010)

[21] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.118.180401 for detailed analytical derivations and discussion of Figs. 2–4, which includes Refs. [30–33]

[22] Born, Max (1926). "Quantenmechanik der Stossvorgänge". Zeitschrift für Physik. 38: 803. Bibcode:1926ZPhy...38..803B. doi:10.1007/BF01397184